On Odd Graceful Labeling of the Generalization of Cyclic Snakes

E. M. Badr
Scientific Computing
Department, Faculty of
Computers and Informatics,
Benha University, Benha
13 518, Egypt
www.bu.edu.eg

ABSTRACT
The objective of this paper is to present a new class of odd graceful graphs. In particular, we show that the linear cyclic snakes (1, *k*) C_4- *snake* and (2, *k*) C_4- *snake* are odd graceful. We prove that the linear cyclic snakes (1, *k*) C_6- *snake* and (2, *k*) C_6- *snake* are odd graceful. We also prove that the linear cyclic snakes (1, *k*) C_8- *snake* and (2, *k*) C_8- *snake* are odd graceful. We generalize the above results "*the linear cyclic snakes* (*m*, *k*) C_4- *snake*, (*m*, *k*) C_6-*snake* and (*m*, *k*) C_8-*snake are odd graceful* ". Finally, we introduce a new conjecture" *All the linear cyclic snakes* (*m*, *k*) C_n-*snake*s *are odd graceful* if *n is even*)".

Keywords
Graph Labeling, Odd Graceful Graphs, Cyclic Snakes

1. INTRODUCTION
The graphs considered here will be finite, undirected and simple. The symbols *V*(*G*) and *E*(*G*) will denote the vertex set and edge set of a graph *G* respectively. *p* and *q* denote the number of vertices and edges of *G* respectively.

A graph *G* of size *q* is odd-graceful, if there is an injection ϕ from *V(G)* to {0, 1, 2, …, 2*q*-1} such that, when each edge *xy* is assigned the label or weight $|\phi(x) - \phi(y)|$, the resulting edge labels are {1, 3, 5, …, 2*q*-1}. This definition was introduced in 1991 by Gnanajothi [1] who proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs.

Several surveys have been written, for instance, Gallian [2] has surveyed graph labeling, including over 1500 articles related to graph labelings. In 2012, Badr and Moussa [3] introduced odd graceful labelings of the kC_4- *snakes* (for the general case), kC_8 *and* kC_{12}- *snakes* (for even case). They also proved that the linear kC_n- *snakes* is odd graceful if and only if *n* and *k* are even. In 2012 Badr [4] show an odd graceful labeling of the linear $kC_4 - snake$ e mK_1 and therefore we introduce the odd graceful labeling of $kC_4 - snake$ e mK_1 (for the general case). He proved that the subdivision of linear $kC_3 - snake$ is odd graceful. He also prove that the subdivision of linear $kC_3 - snake$ with m-pendant edges is odd graceful and he presented an odd graceful labeling of the crown graph P_n e mK_1. In 2013 Badr [5] show that the revised friendship graphs $F(kC_4)$, $F(kC_8)$, $F(kC_{12})$, $F(kC_{16})$ and $F(kC_{20})$ are odd graceful where *k* is any positive integer. He introduced a new conjecture " The revised friendship graph $F(kC_n)$ is odd graceful where *k* is any positive integer and $n = 0$ (*mod* 4). Rosa [6] defined a triangular snake (or Δ -*snake*) as a connected graph in which all blocks are triangles and the block-cut-point graph is a path. Let Δ_k -*snake* be a Δ -*snake* with *k* blocks while $n\Delta_k$ -*snake* be a Δ -*snake* with *k* blocks and every block has *n* number of triangles with one common edge. Badr and Abdel-aal [7] proved that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ -*snake*). They proved that the all subdivision of $2m\Delta_1$ -*snake* are odd graceful. They also generalized the above two results (all subdivision of $2m\Delta_k$ -snake are odd graceful). In 2013 Badr and Abdel-aal [8] show that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ -snake). They also proved that the all subdivision of $2m\Delta_1$-snake are odd graceful and they generalized the above two results (the all subdivision of 2 $m\Delta_k$ -snake are odd graceful). Barrientos [9] generalized the definition of triangular snakes by the following definition.

Definition 1.2
A connected graph in which the *k* blocks are isomorphic to the cycle C_n and the block-cutpoint graph is a path denoted by kC_n-*snake*.

Now, we generalize the definition of kC_n-*snake* by the following definition.

Definition 1.3
The family of graphs consisting of *k* block of C_n with two non-adjacent vertices in common where every block has *m* copies of C_n and the block-cutpoint graph is a path denoted by (*m*, *k*) C_n.

Definition 1.4
The (*m*, *k*) C_n *snake* is called linear, if the block-cut-vertex graph of (*m*, *k*) C_n *snake* has the property that the distance between any two consecutive cut-vertices is $\lfloor n/2 \rfloor$.

Example 1.5

(a) (b)

Figure 1: a) The linear (2, 1) C_4-*snake* and b) The linear (3,2) C_4-*snake*

In this paper, we show that the linear cyclic snakes (1, *k*) C_4-*snake* and (2, *k*) C_4- *snake* are odd graceful. We prove that the linear cyclic snakes (1, *k*) C_6-*snake* and (2, *k*) C_6- *snake* are odd graceful. We also prove that the linear cyclic snakes (1, *k*) C_8- *snake* and (2, *k*) C_8- *snake* are odd graceful. We generalize the above results "*the linear cyclic snakes* (*m*, *k*) C_4- *snake*, (*m*, *k*) C_6-*snake* and (*m*, *k*) C_8-*snake are odd graceful* ". Finally, we introduce a new conjecture" *All the linear cyclic snakes* (*m*, *k*) C_n-*snake*s *are odd graceful* if *n is even*)".

2. MAIN RESULTS

The following theorem was introduced by Badr and Moussa [3] but we can introduce this theorem using a new labeling.

Theorem 2.1: The linear graph (1, *k*) C_4-snake is odd graceful.

Proof: See our technical report [10].

Theorem 2.2: All the linear cyclic snakes (2, *k*) C_4-*snake*s are odd graceful.

Proof:

Let *G* = (2, *k*) C_4-*snake*s has *q* edges and *p* vertices. The graph *G* consists of the vertices (u_1, u_2, …, u_{k+1}) and w_{ij} where *i* = 1, 2, …, *k* and *j* = 1, 2, 3, 4.

We can construct the graph *G* = (2, *k*) C_4-*snakes* as the following:

1-We label the block-cutpoint graph by u_i where *i* = 1, 2,…, *k*+1.

2-We label the vertices which adjacent to u_i and u_{i+1} by w_{ij} where *i* = 1, 2, …, *k* and *j* = 1, 2, 3, 4, as shown in Figure 2.

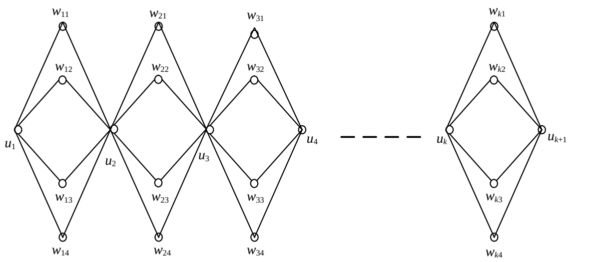

Figure 2: The graph (2, *k*) C_4-snake.

Clearly, the graph *G* = (2, *k*) C_4-*snakes* has *q* = 8*k* edges and *p* = 5*k* + 1 vertices.

We prove that all the linear cyclic snakes (2, *k*) C_4-*snake*s are graceful. Let us consider the following numbering ϕ of the vertices of the graph *G*:

$$\phi(u_i) = 8i - 8 \qquad 1 \le i \le k+1$$
$$\phi(w_{ij}) = 2q - 8i - 2j + 9 \qquad 1 \le i \le k, 1 \le j \le 4$$

a) $\underset{v \in V}{Max}\ \phi(v) = \max\left\{ \max_{1 \le i \le k+1} (8i\text{-}8), \max_{\substack{1 \le j \le 4 \\ 1 \le i \le k}} (2q - 8i - 2j + 9) \right\} = 2q - 1$

, the maximum value of all odd integers. Thus $\phi(v) \in \{0, 1, 2 \ldots, 2q - 1\}$

(b) Clearly ϕ is a one – to – one mapping from the vertex set of G to {0, 1, 2, …, 2*q*-1}.

c) It remains to show that the labels of the edges of *G* are all the odd integers of the interval [1, 2*q*-1].

The range of
$|\ \phi(w_{i1}) - \phi(u_i)\ | = \{2q - 16i + 15 \ :\ 1 \le i \le k\} =$
$\{\ 2q - 1, 2q - 17, ..., 2q - 16k + 15\}$

The range of
$|\ \phi(w_{i1}) - \phi(u_{i+1})\ | = \{2q - 16i + 7 \ :\ 1 \le i \le k\} =$
$\{\ 2q - 9, 2q - 25, ..., 2q - 16k + 7\}$

The range of
$|\ \phi(w_{i2}) - \phi(u_i)\ | = \{2q - 16i + 13 \ :1 \le i \le k\} =$
$\{\ 2q - 3, 2q - 19, ..., 2q - 16k + 13\}$

The range of
$|\ \phi(w_{i2}) - \phi(u_{i+1})\ | = \{2q - 16i + 5 \ :1 \le i \le k\} =$
$\{\ 2q - 11, 2q - 27, ..., 2q - 16k + 5\}$

The range of
$|\ \phi(w_{i3}) - \phi(u_i)\ | = \{2q - 16i + 11 \ :1 \le i \le k\} =$
$\{\ 2q - 5, 2q - 21, ..., 2q - 16k + 11\}$

The range of
$|\ \phi(w_{i3}) - \phi(u_{i+1})\ | = \{2q - 16i + 3 \ :1 \le i \le k\} =$
$\{\ 2q - 13, 2q - 29, ..., 2q - 16k + 3\}$

The range of
$|\ \phi(w_{i4}) - \phi(u_i)\ | = \{2q - 16i + 9 : 1 \le i \le k\} =$
$\{\ 2q - 7, 2q - 23, ..., 2q - 16k + 9\}$

The range of
$|\ \phi(w_{i4}) - \phi(u_{i+1})\ | = \{2q - 16i + 1 : 1 \le i \le k\} =$
$\{\ 2q - 15, 2q - 31, ..., 2q - 16k + 1\}$

Hence {| ϕ (*u*) - ϕ (*v*) | : *uv* ϵ *E* } = {1, 3, …, 2*q*-1}so that the linear (2, *k*) C_4-*snake*s is odd graceful. ■

Example 2.3

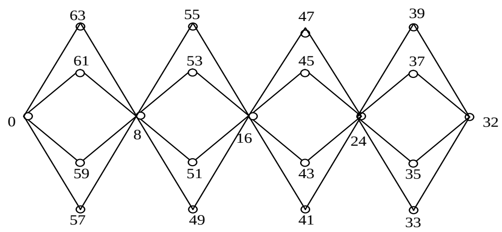

Figure 3: The odd graceful labeling of the linear (2, 4) C_4-snake.

Now, we generalize the above Theorems by the following Theorem.

Theorem 2.4: All the linear cyclic snakes (*m*, *k*) C_4-*snake*s are odd graceful.

Proof:

Let *G* = (*m*, *k*) C_4-*snake*s has *q* edges and *p* vertices. The graph *G* consists of the vertices (u_1, u_2, …, u_{k+1}) and w_{ij} where *i* = 1, 2, …, *k* and *j* = 1, 2, …, 2*m*.

We can construct the graph *G* = (*m*, *k*) C_4-*snakes* as the following:

1- We label the block-cutpoint graph by u_i where *i* = 1, 2, …, *k*+1.

2- We label the vertices which adjacent to u_i and u_{i+1} by w_{ij} where *i* = 1, 2, …, *k* and *j* = 1, 2, …, 2*m*, as shown in Figure 4.

Clearly, the graph G = (2, k) C_4-*snakes* has $q = 2mk$ edges and $p = mk+k+1$ vertices.

We prove that all the linear cyclic snakes (m, k) C_4-*snake*s are odd graceful. Let us consider the following numbering ϕ of the vertices of the graph G:

$$\phi(u_i) = 4m(i-1) \qquad 1 \le i \le k+1$$
$$\phi(w_{ij}) = 2q - 4m(i-1) - 2j + 1 \qquad 1 \le i \le k, 1 \le j \le 2m$$

(a)

$$\underset{v \in V}{Max}\ \phi(v) = \max\left\{ \max_{1 \le i \le k+1} 4m(i-1), \max_{\substack{1 \le j \le 2m \\ 1 \le i \le k}} (2q - 4m(i-1) - 2j + 1) \right\}$$
$$= 2q - 1$$

the maximum value of all odd integers. Thus $\phi(v) \in \{0, 1, 2 \dots, 2q-1\}$

(b) Clearly ϕ is a one – to – one mapping from the vertex set of G to {0, 1, 2, …, 2q-1}.

c) It remains to show that the labels of the edges of G are all the odd integers of the interval [1, 2q-1].

The range of

$$|\ \phi(w_{ij}) - \phi(u_i)\ | =$$
$$\{\ 2q - 8m(i-1) - 2j + 1 \ , 1 \le i \le k, 1 \le j \le 2m\ \}$$

The range of

$$|\ \phi(w_{ij}) - \phi(u_{i+1})\ | =$$
$$\{\ 2q - 4m(2i-1) - 2j + 1 \ , 1 \le i \le k, 1 \le j \le 2m\ \}$$

Hence {| ϕ (u) - ϕ (v) | : $uv \in E$ } = {1, 3,…, 2q-1}so that the all the linear cyclic snakes (m, k) C_4-*snake*s are odd graceful. ■

The following theorem was introduced by Badr and Moussa [3] but we can introduce this theorem using a new labeling.

Theorem 2.5: The linear graph (1, k) C_6 is odd graceful.

Proof: See our technical report [10].

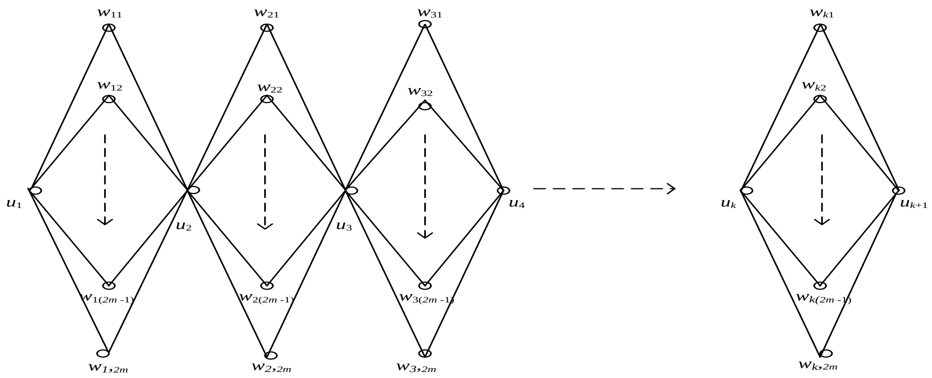

Figure 4: The graph (m, k) C_4-snake.

Theorem 2.6: All linear cyclic snakes (2, k) C_6 are oddiii) graceful.

Proof:

Let G = (2, k) C_6-*snake*s has q edges and p vertices. The graph G consists of the vertices (u_1, u_2, …, u_{k+1}) , w_{ij} where i = 1, 2, …, k and j = 1, 2 , x_{ij} where i = 1, 2, …, 2k and j = 1, 2 and v_{ij} where i = 1, 2, …, k and j = 1, 2

We can construct the graph G = (2, k) C_6-*snakes* as the following:

1- We label the block-cutpoint graph by u_i where i = 1, 2, …, k+1.

ii) 2- We label the vertices which adjacent to u_i and u_{i+1} by w_{ij} where i = 1, 2, …, k and j = 1, 2.

3- We label the vertices which adjacent to u_i and w_{ij} by $x_{(2i-1)j}$ where i = 1, 2, …, k and j = 1, 2.

4- We label the vertices which adjacent to u_{i+1} and w_{ij} by $x_{(2i)j}$ where i = 1, 2, …, k and j = 1, 2.

5- We label the vertices which adjacent to u_i and u_{i+1} by v_{ij} where i = 1, 2, …, k and j = 1, 2. as shown in Figure 5.

Clearly, the graph G = (2, k) C_6-*snakes* has $q = 16k$ edges and $p = 13k + 1$ vertices.

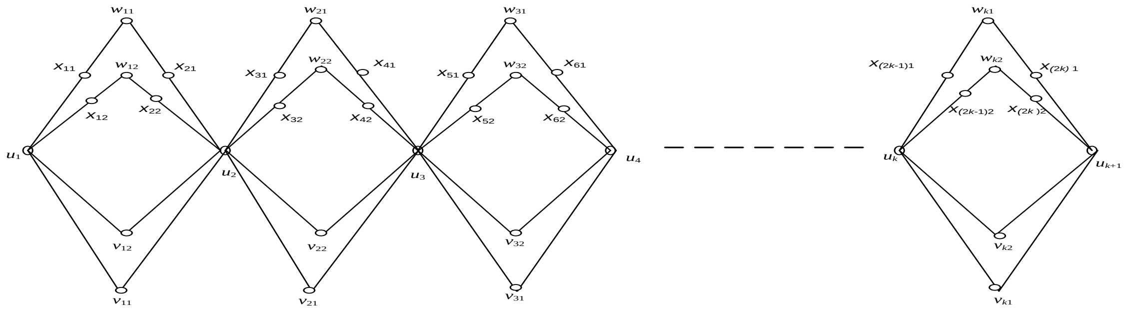

Figure 5: The linear (2, k) C_6-snakes

We prove that all the linear cyclic snakes (2, k) C_8-*snake*s are odd graceful. Let us consider the following numbering ϕ of the vertices of the graph G:

$\phi(u_i) = 8(i - 1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_{ij}) = 8i - 4j + 2$, $i = 1, 2, 3\ldots k$, $j = 1, 2$

$\phi(x_{ij}) = 2q - 4i - 2j + 5$, $i = 1, 2, 3\ldots 2k$ for all $j = 1, 2$

$\phi(v_{ij}) = 8k - 8i + 2j - 1$, $i = 1, 2, 3\ldots k$, $j = 1, 2$

a) $$\underset{v\in V}{Max}\ \phi(v) = \max\left\{\max_{1\le i\le k+1} 8(i\text{-}1), \max_{\substack{1\le j\le 2\\1\le i\le k}}(8i - 4j + 2), \max_{\substack{1\le j\le 2\\1\le i\le 2k}}(2q - 4i - 2j + 5), \max_{\substack{1\le j\le 2\\1\le i\le k}}(8(k-i) + 2j - 1)\right\} =$$

$2q - 1$, the maximum value of all odd integers. Thus $\phi(v) \in \{0, 1, 2 \ldots, 2q-1\}$.

(b) Clearly ϕ is a one – to – one mapping from the vertex set of G to $\{0, 1, 2, \ldots, 2q-1\}$.

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1, 2q-1]$.

The range of
$|\ \phi(x_{(2i)j}) - \phi(u_{i+1})\ | = \{\ 2q - 16i - 2j + 5\ , 1\le i\le k, 1\le j\le 2\ \}$
The range of
$|\ \phi(x_{(2i)j}) - \phi(w_{ij})\ | = \{\ 2q - 16i + 2j + 3\ , 1\le i\le k, 1\le j\le 2\ \}$
The range of
$|\ \phi(x_{(2i-1)j}) - \phi(u_i)\ | = \{\ 2q - 16i - 2j + 17\ , 1\le i\le k, 1\le j\le 2\ \}$
The range of
$|\ \phi(x_{(2i-1)j}) - \phi(w_{ij})\ | = \{\ 2q - 16i + 2j + 7\ , 1\le i\le k, 1\le j\le 2\ \}$
The range of
$|\ \phi(v_{ij}) - \phi(u_i)\ | = \{\ |8k - 16i + 2j + 7|\ , 1\le i\le k, 1\le j\le 2\ \}$
The range of
$|\ \phi(v_{ij}) - \phi(u_{i+1})\ | = \{\ |8k - 16i + 2j - 1|\ , 1\le i\le k, 1\le j\le 2\ \}$

Hence $\{|\ \phi(u) - \phi(v)\ | : uv \in E\} = \{1, 3, \ldots, 2q-1\}$ so that the linear (2, k) C_6-*snake*s is odd graceful. ■

Theorem 2.7: All the linear cyclic snakes (m, k) C_6-*snake*s are odd graceful.

Proof:

Let G = (m, k) C_6-*snake*s has q edges and p vertices. The graph G consists of the vertices $(u_1, u_2, \ldots, u_{k+1})$, w_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$, x_{ij} where $i = 1, 2, \ldots, 2k$ and $j = 1, 2, \ldots, m$ and v_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$

We can construct the graph G = (m, k) C_8-*snakes* as the following:

1- We label the block-cutpoint graph by u_i where $i = 1, 2, \ldots, k+1$.

2- We label the vertices which adjacent to u_i and u_{i+1} by w_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$

3- We label the vertices which adjacent to u_i and w_{ij} by $x_{(2i-1)j}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$.

4- We label the vertices which adjacent to u_{i+1} and w_{ij} by $x_{(2i)j}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$.

5- We label the vertices which adjacent to u_i and u_{i+1} by v_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$. as shown in Figure 6.

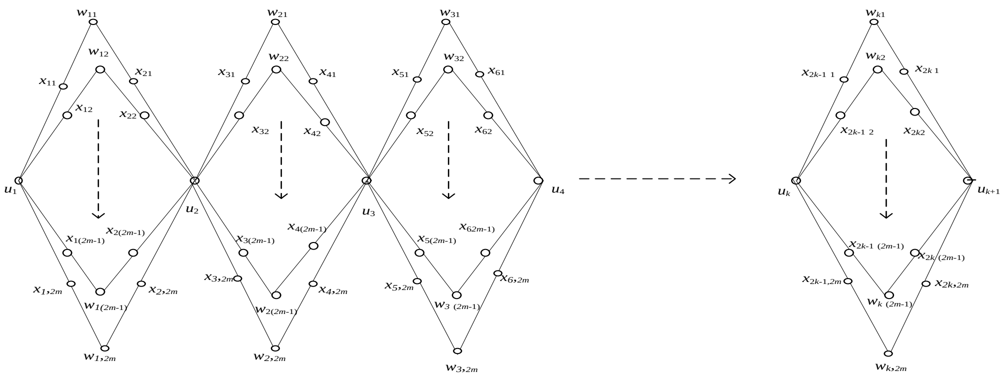

Figure 6: The graphs (m, k) C_8-snake

Clearly, the graph G = (m, k) C_6-*snakes* has $q = 8mk$ edges. and $p = (6m+1)k + 1$ vertices. We prove that all the linear cyclic snakes (m, k) C_8-*snake*s are odd graceful. Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi(u_i) = 4m(i - 1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_{ij}) = 8mi - 4j + -4m+2$, $i = 1, 2, 3\ldots k$, $j = 1, 2, \ldots, m$.

$\phi(x_{ij}) = 2q - 2m(i-1) - 2j + 1$, $i = 1, 2, \ldots 2k$ for all $j = 1, 2, \ldots, m$.

$\phi(v_{ij}) = 4mk - 4mi + 2j - 1$, $i = 1, 2, 3\ldots k$, $j = 1, 2, \ldots, m$.

a) $$\underset{v\in V}{Max}\ \phi(v) = \max\left\{\max_{1\le i\le k+1} 4m(i\text{-}1), \max_{\substack{1\le j\le m\\1\le i\le k}} 8mi - 4j - 4m + 2, \max_{\substack{1\le j\le m\\1\le i\le k}} 2q - 2m(i\text{-}1) - 2j + 1, \max_{\substack{1\le j\le m\\1\le i\le k}} 4m(k-i) + 2j + 1\right\} =$$

$= 2q - 1$, the maximum value of all odd integers. Thus $\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

(b) Clearly ϕ is a one –to– one mapping from the vertex set of G to $\{0, 1, 2, \ldots, 2q-1\}$.

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1, 2q-1]$.

The range of
$|\ \phi(x_{(2i)j}) - \phi(u_{i+1})| = \{\ 2q - 8mi + 2m - 2j + 1\ \ , 1 \le i \le k, 1 \le j \le m\ \}$
The range of
$|\ \phi(x_{(2i)j}) - \phi(w_{ij})| = \{\ 2q - 12mi + 6m + 2j - 1\ \ , 1 \le i \le k, 1 \le j \le m\ \}$
The range of
$|\ \phi(x_{(2i-1)j}) - \phi(u_i)| = \{\ 2q - 8m(i+1) - 2j + 1\ , 1 \le i \le k, 1 \le j \le m\}$
The range of
$|\ \phi(x_{(2i-1)j}) - \phi(w_{ij})| = \{\ 2q - 12mi + 8m + 2j - 1\ \ , 1 \le i \le k, 1 \le j \le m\ \}$
The range of
$|\ \phi(v_{ij}) - \phi(u_i)| = \{\ |4mk - 8mi + 4m + 2j - 1|\ \ , 1 \le i \le k, 1 \le j \le m\ \}$
The range of
$|\ \phi(v_{ij}) - \phi(u_{i+1})| = \{\ |4mk - 8mi + 2j - 1|\ , 1 \le i \le k, 1 \le j \le m\ \}$

Hence $\{|\ \phi\ (u) - \phi\ (v)\ | : uv \in E\ \} = \{1, 3, \ldots, 2q-1\}$ so that the linear (m, k) C_6-*snake*s is odd graceful. ■

The following theorem was introduced by Badr and Moussa [3] but we can introduce this theorem using a new labeling.

Theorem 2.8: The linear $(1, k)$ C_8-snakes is odd graceful.

Proof: See our technical report [10].

Theorem 2.9: All the linear cyclic snakes $(2, k)$ C_8-*snake*s are graceful.

Proof:

Let $G = (2, k)$ C_8-*snake*s has q edges and p vertices. The graph G consists of the vertices $(u_1, u_2, \ldots, u_{k+1})$, w_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, 3, 4$ and x_{ij} where $i = 1, 2, \ldots, 2k$ and $j = 1, 2, 3, 4$.

We can construct the graph $G = (2, k)$ C_8-*snakes* as the following:

1- We label the block-cutpoint graph by u_i where $i = 1, 2, \ldots, k+1$.

2- We label the vertices which adjacent to u_i and u_{i+1} by w_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, 3, 4$.

3- We label the vertices which adjacent to u_i and w_{ij} by $x_{(2i-1)j}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, 3, 4$.

4- We label the vertices which adjacent to u_{i+1} and w_{ij} by $x_{(2i)j}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, 3, 4$, as shown in Figure 7.

Clearly, the graph $G = (2, k)$ C_4-*snakes* has $q = 16k$ edges and $p = 13k + 1$ vertices.

We prove that all the double cyclic snakes $(2, k)$ C_8-*snake*s are odd graceful. Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi(u_i) = 16(i-1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_{ij}) = 16i - 4j + 2$, $i = 1, 2, 3\ldots k$, $j = 1, 2, 3, 4$

$\phi(x_{ij}) = 2q - 8i - 2j + 9$, $i = 1, 2, 3\ldots 2k$ for all $1 \le j \le 4$

(a)

$$\underset{v \in V}{Max}\ \phi(v) = \max\left\{ \max_{1 \le i \le k+1} 16(i-1), \max_{\substack{1 \le j \le 4 \\ 1 \le i \le k}} (16i - 4j + 2), \max_{\substack{1 \le j \le 4 \\ 1 \le i \le 2k}} (2q - 8i - 2j + 9) \right\} = 2q - 1$$

, the maximum value of all odd integers. Thus $\phi(v) \in \{\ 0, 1, 2\ \ldots, 2q-1\ \}$.

(b) Clearly ϕ is a one – to – one mapping from the vertex set of G to $\{0, 1, \ldots, 2q-1\}$.

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1, 2q-1]$.

The range of
$|\ \phi(x_{(2i)j}) - \phi(u_{i+1})| = \{\ 2q - 32i - 2j + 9\ \ , 1 \le i \le k, 1 \le j \le 4\ \}$
The range of
$|\ \phi(x_{(2i)j}) - \phi(w_{ij})| = \{\ 2q - 32i + 2j + 7\ \ , 1 \le i \le k, 1 \le j \le 4\ \}$
The range of
$|\ \phi(x_{(2i-1)j}) - \phi(u_i)| = \{\ 2q - 32i - 2j + 33\ \ , 1 \le i \le k, 1 \le j \le 4\ \}$
The range of
$|\ \phi(x_{(2i-1)j}) - \phi(w_{ij})| = \{\ 2q - 32i + 2j + 15\ \ , 1 \le i \le k, 1 \le j \le 4\ \}$

Hence $\{|\ \phi\ (u) - \phi\ (v)\ | : uv \in E\ \} = \{1, 3, \ldots, 2q-1\}$ so that the linear $(2, k)$ C_8-*snake*s is odd graceful.
■

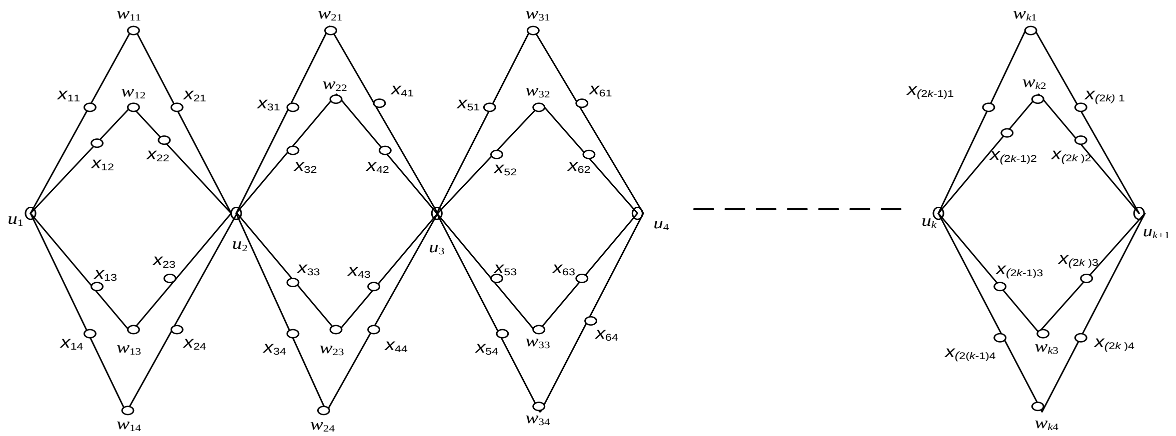

Figure 7: The graph (2, *k*) C_8-snakes

Theorem 2.10: All the linear cyclic snakes (m, k) C_8-*snake*s are odd graceful.

Proof:

Let $G = (m, k)$ C_8-*snake*s has q edges and p vertices. The graph G consists of the vertices $(u_1, u_2, \ldots, u_{k+1})$, w_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, 2m$ and x_{ij} where $i = 1, 2, \ldots, 2k$ and $j = 1, 2, \ldots, 2m$.

We can construct the graph $G = (m, k)$ C_8-*snakes* as the following:

1- We label the block-cutpoint graph by u_i where $i = 1, 2, \ldots, k+1$.

2- We label the vertices which adjacent to u_i and u_{i+1} by w_{ij} where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, 2m$.

3- We label the vertices which adjacent to u_i and w_{ij} by $x_{(2i-1)j}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, 2m$.

4- We label the vertices which adjacent to u_{i+1} and w_{ij} by $x_{(2i)j}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, 3, 4$, as shown in Figure8.

Clearly, the graph $G = (m, k)$ C_4-*snakes* has $q = 8mk$ edges. and $p = (6m+1)k + 1$ vertices. We prove that all the double cyclic snakes (m, k) C_8-*snake*s are odd graceful. Let us consider the following numbering ϕ of the vertices of the graph G:

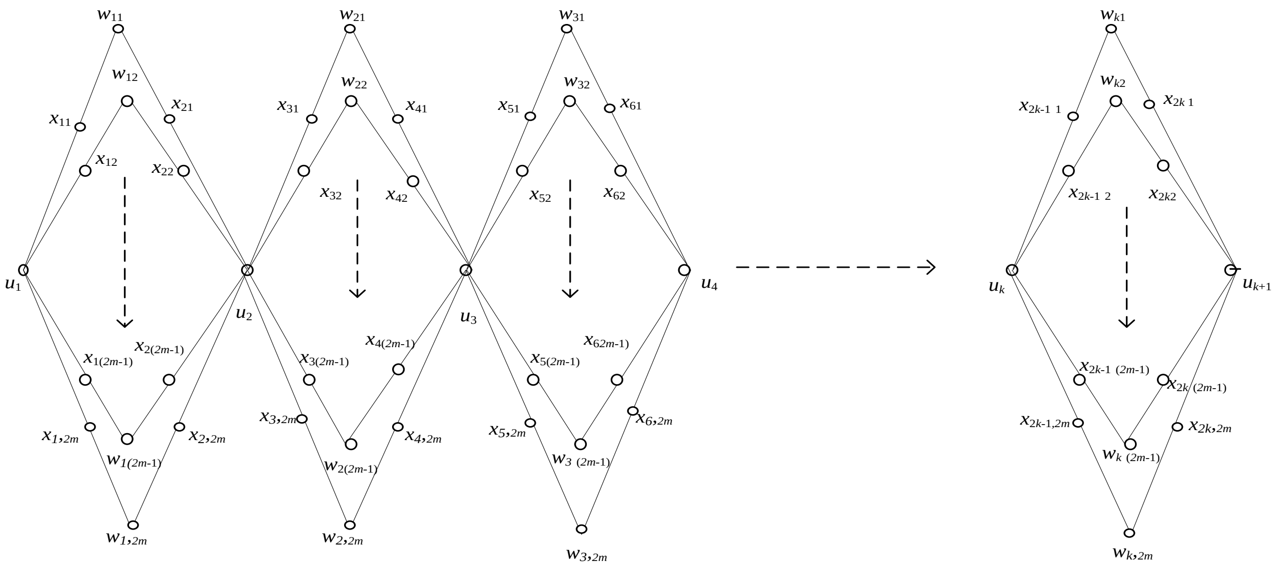

Figure 8: The graphs (m, k) C_8-snake

$\phi(u_i) = 8m(i-1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_{ij}) = 8mi - 4j + 2$, $i = 1, 2, 3\ldots k$; $j = 1, 2, 3\ldots 2m$

$\phi(x_{ij}) = 2q - 4m(i-1) - 2j + 1$, $i = 1, 2, 3\ldots 2k$; $j = 1, 2, 3\ldots 2m$

(a)

$$\underset{v\in V}{Max}\ \phi(v) = \max\left\{\max_{1\le i\le k+1} 8m(i-1), \max_{\substack{1\le j\le 2m\\1\le i\le k}} 8mi - 4j + 2, \max_{\substack{1\le j\le 2m\\1\le i\le 2k}} (2q - 4m(i-1) - 2j + 1)\right\} = 2q - 1$$

the maximum value of all odd integers. Thus $\phi(v) \in \{0, 1, 2 \ldots 2q-1\}$

(b) Clearly ϕ is a one –to– one mapping from the vertex set of G to $\{0, 1, 2, \ldots, 2q\text{-}1\}$.

c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1, 2q\text{-}1]$.

The range of

$$|\ \phi(x_{(2i)j}) - \phi(u_{i+1})\ | = \{\ 2q - 16mi + 4m - 2j + 1, 1\le i\le k, 1\le j\le 2m\ \}$$

The range of

$$|\ \phi(x_{(2i)j}) - \phi(w_{ij})\ | = \{\ 2q - 16mi + 12m - 2j + 1\ , 1\le i\le k, 1\le j\le 2m\ \}$$

The range of

$$|\phi(x_{(2i-1)j}) - \phi(u_i)\ | = \{\ 2q - 16mi + 16m - 2j + 1\ , 1\le i\le k+1, 1\le j\le 2m\ \}$$

The range of

$$|\ \phi(x_{(2i-1)j}) - \phi(w_{ij})\ | = \{\ 2q - 16mi + 8m + 2j - 1\ , 1\le i\le k, 1\le j\le 2m\ \}$$

Hence $\{|\ \phi(u) - \phi(v)\ | : uv \in E\} = \{1, 3, \ldots, 2q\text{-}1\}$ so that the all the linear cyclic snakes (m, k) C_8-*snake*s are odd graceful.■

Conjecture 2.10: All the linear cyclic snakes (m, k) C_n-*snake*s are odd graceful if n even**.**

3. CONCLUSION

In this paper, we show that the linear cyclic snakes $(1, k)$ C_4-*snake* and $(2, k)$ C_4- *snake* are odd graceful. We proved that the linear cyclic snakes $(1, k)$ C_6- *snake* and $(2, k)$ C_6- *snake* are odd graceful. We also proved that the linear cyclic snakes $(1, k)$ C_8- *snake* and $(2, k)$ C_8- *snake* are odd graceful. We generalized the above results "*the linear cyclic snakes* (m, k).

4. ACKNOWLEDGMENTS

This research was supported by the Research Unit of Benha University, Egypt. The author would like to thank the reviewers of this paper for their valuable comments.

5. REFERENCES

[1] R.B. Gnanajothi, Topics in graph theory, Ph.D. thesis, Madurai Kamaraj University, India, 1991.

[2] J. A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, (http://www.combinatorics.org/) DS 16 (2013).

[3] E. M. Badr and M. I. Moussa, ODD GRACEFUL LABELINGS OF CYCLIC SNAKES , Electronic Journal of Nonlinear Analysis and Application, Vol. 6, December 2012.

[4] E. M. Badr (2014), On the Odd Gracefulness of Cyclic Snakes With Pendant Edges**,** International journal on applications of graph theory in wireless ad hoc networks and sensor networks (GRAPH-HOC) Vol.4, No.4, December 2012

[5] E. M. Badr (2014), Odd Graceful Labeling of the revised friendship graphs, International Journal of Computer Applications (0975 – 8887) Volume 65– No.11, March 2013

[6] A. Rosa, Cyclic Steiner Triple Systems and Labeling of Triangular Cacti, Scientia, 5 (1967) 87-95.

[7] E. M. Badr and M. E. Abdel-aal, Odd Graceful Labeling for the Subdivision of Double Triangles Graphs, International Journal of Soft Computing, Mathematics and Control (IJSCMC), Vol.2, No.1, February 2013.

[8] E. M. Badr and M. E. Abdel-aal, (2013), ODD GRACEFULL LABELING FOR THE SUBDIVISON OF DOUBLE TRIANGLES GRAPHS, International Journal of Soft Computing, Mathematics and Control (IJSCMC), Vol.2, No.1, February 2013

[9] Christian Barrientos, Graceful labelings of cyclic snakes, Ars Combinatorica 60 (2001), pp. 85-96.

[10] E. M. Badr (2014), Complete Reference for Odd Graceful Labeling of Cyclic Snakes,Technical Report 2, 2014, Benha Unveristy, Faculty of Computers and Informatics, http://www.bu.edu.eg/staff/alsayedbadr7